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# Decomposition of weather forecast error using rearrangements of functions

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**Abstract**

A new approach to evaluation of weather forecast error is discussed. Following recent ideas of Hoffman *et al* [11], we decompose the error into a combination of displacement and difference in qualitative features. We give a precise mathematical formulation of this idea using rearrangements of functions, and indicate the work required to justify the formulation. Finally we consider how these ideas may be implemented numerically, indicate some weaknesses of the proposed strategy, and outline an alternative approach.

## 1 Introduction

There have been many attempts to quantify weather forecast error. In this report we consider a new approach to the problem of evaluating the error of a forecast quantity compared to the actual distribution. Most weather forecast errors are simply displacements of significant weather in space or time, such as a rainband arriving a few hours later than expected. Conventional forecast error scores, such as root mean square errors, have limited use as they reward conservative forecasts. For example a forecast which is very good except for misplacing the location of a small intense storm by only a few grid lengths may have a poorer score than one that only forecasts a weak storm in that area. A more informative assessment of the two forecasts is that both have some positioning errors, but the latter has failed to capture the correct qualitative features. Following the pioneering work of Hoffman *et al* [11], we describe a forecast error which is split into two parts: the error due to displacement, and the error due to differences in qualitative features. This measure of error would favour the former forecast in our example above; moreover the two error values (one corresponding to displacement and one to difference in qualitative features) would be descriptive of how the forecasts had failed.

The idea of forecast error decomposition is as follows. Find a displaced version of the actual distribution which is a best fit to the forecast. Use a conventional forecast error score (such as root mean square error) to evaluate the error between the displaced true distribution and the forecast: this value represents the error due to difference in qualitative features. Now calculate a geodesic distance on the set of allowable displacements between the true distribution and the displaced true distribution. This value measures the displacement error. The total error is the sum of the two. For a hypothetical example, we give a sketch of the displaced true distribution which most closely matches the given forecast in Figure 1. Essentially we identify similar qualitative features in the forecast and actual distribution.

In Section 2 we give a mathematical formulation of the above idea using the concept of rearrangement of functions. We must show that this formulation is well posed: the extent to which this has been done, and the work which remains, is detailed in Section 3. In Section

4 we discuss numerical implementation of error forecast decomposition, and also note some weaknesses in the idea.

## 2 A formulation of forecast error decomposition

Consider an important quantity in weather forecasting which is a function of spatial variables, e.g. rainfall, and suppose we have a distribution predicted by a model, and a distribution derived from observations, which we assume to be the true distribution. Hoffman *et al* [**11**] proposed the idea of a forecast error decomposition. They split the difference between forecast and analysis into a combination of displacement error, amplitude error and residual error. The displacement error measures how much of the forecast error can be accounted for by moving the forecast to best fit the analysis, amplitude error measures the change in amplitude of the displaced forecast which best fits the analysis, and the residual error is the remainder. A simple displacement of the forecast was defined to be the composition of the forecast with a transformation which could be any combination of translation, rotation and stretching without tearing. However there is no guarantee that we can find a simple displacement (as defined above) which best fits the analysis. Moreover the difficulties associated with non-uniqueness of the "best fit" were not considered by Hoffman *et al.* Displacement error was calculated as the root mean square of the difference between the displaced forecast and the actual forecast. However if the non-zero values of the displaced forecast are disjoint from the non-zero values of the actual forecast, this will always give a displacement error of twice the root mean square of the actual forecast, regardless of the distance between similar features of the displaced and actual forecast. A strategy based on calculating a geodesic distance on the space of allowable displacements will give values which better match the intuitive idea of displacement error.

Following the ideas of Hoffman et al, we give a formulation of decomposing the forecast error into a combination of displacement and difference in qualitative features. (We discard the concept of residual error.) In order to state the problem precisely, we use the concept of rearrangement of functions from classical analysis. We recall the concept and some properties of rearrangement of functions in the next two paragraphs.

We give an intuitive idea of when a function is a rearrangement of another function: the reader is referred to Cullen and Douglas [**9**] for further details and examples. Let $\Omega$ be a bounded region in $\mathbb{R}^n$ (typically $\mathbb{R}^2$ or $\mathbb{R}^3$), and let $f : \Omega \to \mathbb{R}^d$ be a function. Imagine $\Omega$ is a continuum of infinitesimal particles. Suppose we exchange the particle positions, with each particle retaining its value of $f$. This yields a new function $g$, which is a *rearrangement of* $f$. The idea of rearranging a function can be applied to both scalar and vector valued functions. In our current context, the latter is important if we wish to consider errors in the forecast of a vector valued quantity, such as wind velocity.

A function $f$ has more than one rearrangement (unless it is constant). (See Cullen and Douglas [**9**] for examples of different rearrangements of the identity function defined on the unit interval.) We write $R(f)$ to denote the *set of rearrangements* of a prescribed function $f$. Note that if $f$ and $g$ are rearrangements, then $R(f)$ and $R(g)$ are the same set (because rearrangement is an equivalence relation).

We introduce some terminology. Let $\Omega \subset \mathbb{R}^n$ be bounded. A function $f : \Omega \to \mathbb{R}^d$ is *integrable* if $\int_\Omega |f|_d < \infty$, where $|.|_d$ denotes (Euclidean) distance on $\mathbb{R}^d$. $f$ is *square integrable*, or $f \in L^2(\Omega, \mathbb{R}^d)$, if $\int_\Omega |f|_d^2 < \infty$. We can define a norm on $L^2(\Omega, \mathbb{R}^d)$ by $||f||_2 = \{\int_\Omega |f|_d^2\}^{1/2}$. If we divide by the size of $\Omega$, this quantity is sometimes referred to as the *root mean square* of $f$. Square integrable functions are integrable, but the converse may be false.

We now give a mathematical formulation of the idea of forecast error decomposition. Let $q : \Omega \to \mathbb{R}^d$, where $\Omega \subset \mathbb{R}^n$ is bounded, denote an important quantity in weather forecasting e.g. rainfall. Let $q_1$ be the distribution predicted by the model, $q_2$ the observed (true) distribution. We make the physically reasonable assumption that $q_1$ and $q_2$ are square integrable.

In a conventional approach, the forecast error is $||q_1 - q_2||_2$. In our proposed approach we split the error as follows:
(i) Find $\hat{q} \in R(q_2)$ such that $\hat{q}$ attains

$$\inf_{q \in R(q_2)} ||q_1 - q||_2 \tag{1}$$

(ii) Calculate a geodesic distance between $\hat{q}$ and $q_2$ in $R(q_2)$. This will give a value (2).

(1) represents the error due to difference in qualitative features, whilst (2) represents the error due to displacement. The total error is the sum of the two. (It may be appropriate to calculate the total error as a weighted sum of (1) and (2). For simplicity we do not consider this in this report.)

The above formulation needs to be justified. (1) is a well defined finite number. We need to show that such a $\hat{q} \in R(q_2)$ exists, and establish criteria for it to be unique. If $\hat{q}$ is not unique, the geodesic distance has to be calculated for every minimiser, with the displacement error being the infimum of these values. Finally we note that we have to give a precise definition of the geodesic distance between two rearrangements. These problems are considered in the next section.

## 3 Mathematical results and description of outstanding work

In this section we state and prove results which go some way to justifying the formulation of error decomposition given in the previous section. We indicate the work that remains, and suggest strategies to solve these problems. Before stating our results, we need to review some further facts about rearrangements of functions.

For a non-negative integrable scalar function $f$, defined on the unit interval, there is an (essentially) unique element of $R(f)$ which is an increasing function. We call this element the *increasing rearrangement of* $f$, and denote it $f^*$. The reader is referred to Cullen and Douglas [**9**] for an example. Increasing rearrangements satisfy certain inequalities, a fact that we exploit to prove Proposition 1 below. For non-negative square integrable functions $f$ and $g$ defined on the unit interval, Crowe, Zweibel and Rosenbloom [**8**] showed that

$$||f^* - g^*||_2 \leq ||f - g||_2. \tag{2}$$

An idea which is useful for relating rearrangements is that of measure-preserving mappings. The *measure* of a set indicates its size. There are many possible choices: in $\mathbb{R}$, the measure could be length, in $\mathbb{R}^2$, area, in $\mathbb{R}^3$, volume. Let $\Omega_1 \subset \mathbb{R}^d$ and $\Omega_2 \subset \mathbb{R}^n$ be bounded sets with $\mu(\Omega_1) = \nu(\Omega_2)$, where $\mu$ and $\nu$ denote measures on $\mathbb{R}^d$ and $\mathbb{R}^n$ respectively. A mapping $s : \Omega_1 \to \Omega_2$ is a *measure-preserving mapping* if for each (measurable) set $U \subset \Omega_2$, $\mu(\{x : s(x) \in U\}) = \nu(U)$. Halmos [**10**, page 163, Theorem C] shows that this is equivalent to requiring that for every integrable function $f : \Omega_2 \to \mathbb{R}$ we have

$$\int_{\Omega_1} f \circ s d\mu = \int_{\Omega_2} f d\nu.$$

Such mappings are onto but need not be smooth, nor one to one. Examples of measure-preserving maps include the trajectory mapping for an incompressible fluid. If a measure-preserving mapping is one to one and maps measurable sets to measurable sets, then the inverse of $s$ exists and is also a measure-preserving mapping. Such an $s$ is called a *measure-preserving transformation.* Measure-preserving mappings and transformations preserve (scalar or vector valued) rearrangements in the sense that if $f \in R(g)$ and $s$ is a measure-preserving mapping, then the composition of $f$ with $s$, written $f \circ s$, belongs to $R(g \circ s)$. If $s$ is a measure-preserving transformation, then the reverse implication holds. In the particular case when $\Omega_1 = \Omega_2$, then $f \circ s \in R(f)$.

The concept of measure-preserving transformation allows us to extend the notion of increasing rearrangement to functions defined on sets of finite size. Let $f$ be an integrable scalar valued function defined on a bounded domain $\Omega \subset \mathbb{R}^n$, and let $\tau : [0, \mu(\Omega)] \to \Omega$ be a measure-preserving transformation, where $\mu$ is a measure on $\mathbb{R}^n$. The composition of $f$ with $\tau$ is a function defined on $[0, \mu(\Omega)]$. The increasing rearrangement of this function is called the *increasing rearrangement of* $f$, and is denoted $f^*$. Any other rearrangement may be obtained from the increasing rearrangement by composition with an appropriate measure-preserving mapping. For $\Omega$ as above, Ryff [**14**] proved that for an integrable $f : \Omega \to \mathbb{R}$, there exists a measure-preserving mapping $s : \Omega \to [0, \mu(\Omega)]$ such that $f = f^* \circ s$. Such a composition of functions is called a *polar factorisation of* $f$. This result is not true if we substitute a general rearrangement of $f$ for $f^*$: see Cullen and Douglas [**9**] for a counterexample.

### 3.1 A characterisation of minimising rearrangements

In our formulation of error decomposition given in the previous section, we needed to find $\hat{q} \in R(q_2)$ that attained $\inf_{q \in R(q_2)} ||q_1 - q||_2$. For scalar valued functions, under the assumption that $q_1$ and $q_2$ are non-negative and square integrable, we show that such a $\hat{q}$ may be found. These assumptions are reasonable when $q$ represents a quantity such as rainfall. We also establish a necessary and sufficient condition for the minimiser to be unique, and note a special case when this condition is always satisfied. Finally we conjecture the extent to which minimisers are determined, and give an example which demonstrates that there may be infinitely many minimisers.

**Proposition 1** *Let $q_1$, $q_2 : \Omega \to \mathbb{R}$ be non-negative square integrable functions, where $\Omega \subset \mathbb{R}^n$ is bounded. Then*
*(i) there exists $\hat{q} \in R(q_2)$ such that $\hat{q}$ attains $\inf_{q \in R(q_2)} ||q_1 - q||_2$, and*
*(ii) $\hat{q}$ is the unique minimiser of $||q_1 - q||_2$ over $q \in R(q_2)$ if and only if $\hat{q} = \varphi \circ q_1$ for some increasing function $\varphi$.*

**Proof** (i) $q_1$ has a polar factorisation $q_1 = q_1^* \circ s$ where $s$ is a measure-preserving mapping. Let $q \in R(q_2)$. Now

$$\begin{aligned} ||q_1 - q||_2 &\geq ||q_1^* - q_2^*||_2 \quad (3) \\ &= ||q_1^* \circ s - q_2^* \circ s||_2 \quad (4) \\ &= ||q_1 - q_2^* \circ s||_2 \end{aligned}$$

By way of explanation, (3) follows from Crowe, Zweibel and Rosenbloom [**8**, Corollary 1], and the fact that $s$ is a measure-preserving mapping yields (4). It follows that $\hat{q} = q_2^* \circ s$ attains $\inf_{q \in R(q_2)} ||q_1 - q_2||_2$.

(ii) From (i) we see that $\hat{q}$ attains $\inf_{q \in R(q_2)} ||q_1 - q||_2$ if and only if $||q_1 - \hat{q}||_2 = ||q_1^* - q_2^*||_2$. Noting that the $L^2$ norm is preserved under rearrangement, it follows that $\hat{q}$ attains $\inf_{q \in R(q_2)} ||q_1 - q||_2$ if and only if

$$\int_\Omega q_1 \hat{q} = \int_0^{\mu(\Omega)} q_1^* q_2^*, \tag{5}$$

where $\mu(\Omega)$ denotes the measure of $\Omega$. Burton [**6**, Theorems 1, 3 and 5] yields that $\hat{q}$ is the unique element of $R(q_2)$ where equality holds in (5) if and only if $\hat{q} = \varphi \circ q_1$ for some increasing function $\varphi$. This completes the proof.

**Remark** If $q_1$ has no level sets of positive measure, that is the sets $L_\alpha = \{x \in \Omega : q_1(x) = \alpha\}$ have zero size for each $\alpha \in \mathbb{R}$, then we can find an increasing function $\varphi$ such that $\varphi \circ q_1$ is a rearrangement of $q_2$. Proposition 1 (ii) yields that this is the unique minimiser of $||q_1 - q||_2$ over $q \in R(q_2)$.

**Conjecture** Minimisers of $||q_1 - q||_2$ over $q \in R(q_2)$ are uniquely determined up to level sets of $q_1$ with positive measure, and when two minimisers are restricted to any level set of $q_1$ with positive measure, they are rearrangements on that level set.

We consider an example which illustrates that the minimiser may not be unique. Let $q_1, q_2 : [-1,1]^2 \to \mathbb{R}$ be respectively the forecast and actual distribution of some meteorological quantity, where

$$q_1(x,y) = \begin{cases} 3 & \text{if } x^2 + y^2 \leq 1/16, \\ 2 & \text{if } 1/16 < x^2 + y^2 \leq 9/16, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$q_2(x,y) = \begin{cases} 3 & \text{if } x^2 + y^2 \leq 1/4, \\ 0 & \text{otherwise.} \end{cases}$$

Further define

$$q_3(x,y) = \begin{cases} 3 & \text{if } x^2 + y^2 \leq 1/16, \text{ or } 6/16 \leq x^2 + y^2 \leq 9/16, \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that $q_3 \in R(q_2)$, and moreover $||q_1 - q_2||_2 = ||q_1 - q_3||_2 = \inf_{q \in R(q_2)} ||q_1 - q||_2$. In fact any $\tilde{q} \in R(q_2)$ which satisfies

$$\{(x,y) : x^2 + y^2 \leq 1/16\} \subset \{(x,y) : \tilde{q}(x,y) = 3\} \subset \{(x,y); x^2 + y^2 \leq 9/16\}$$

will be a minimiser. In fact for most choices of piecewise constant functions $q_1$ and $q_2$, there will be infinitely many $\tilde{q} \in R(q_2)$ which attain $\inf_{q \in R(q_2)} ||q_1 - q||_2$.

### 3.2 Approaches to calculating displacement error

To find the error due to displacement, we need to calculate a "distance" between two functions $\hat{q}$ and $q_2$ which are rearrangements of each other. We discuss two approaches to this problem: both follow the work of Brenier. In the first we seek a velocity field which will transport $\hat{q}$ to $q_2$, and then calculate its kinetic energy. The second approach is to exploit Brenier's work ([**2**] and [**4**]) on the distance between measure-preserving mappings directly; to use these results we require a suitable factorisation of $\hat{q}$ and $q_2$. Existing theory is insufficient to show that either strategy is well posed. We state both approaches and indicate the work which remains; it is not clear which problem will be easier to solve. In this subsection we have the example of $q$ representing rainfall in mind: we assume $\hat{q}, q_2 : \Omega \to \mathbb{R}$ where $\Omega$ is a bounded domain in $\mathbb{R}^n$.

**Approach 1: Transport $\hat{q}$ to $q_2$ using an incompressible velocity field, and calculate the Least Action**

We wish to find the geodesic distance between $\hat{q}$ and $q_2$ in $R(q_2)$. We wish to consider "paths" which connect $\hat{q}$ and $q_2$: such trajectories $g : \Omega \times [0,1] \to \mathbb{R}$ will satisfy $g(.,0) = \hat{q}$, $g(.,1) = q_2$ and $g(.,t) \in R(q_2)$ for each $t \in [0,1]$. Suppose we can find an incompressible velocity field that transports $\hat{q}$ to $q_2$: we call the solution of the transport equation a *path between $\hat{q}$ and $q_2$*. (Note that such a path satisfies $g(.,t) \in R(q_2)$ for each $t \in [0,1]$.) Brenier [**5**] has shown the existence of such a velocity field for some specific examples of $\hat{q}$ and $q_2$. For a path $g$ define an action by

$$A(g) = \frac{1}{2} \int_0^1 \int_\Omega |\partial_t g(t,x)|^2 dx dt. \tag{6}$$

Define the action of the shortest path between $\hat{q}$ and $q_2$ to be the infimum of (6) over all possible paths (as defined above). This quantity is the geodesic distance between $\hat{q}$ and $q_2$ in $R(q_2)$. It remains to show that we can always find an incompressible velocity field that transports $\hat{q}$ to $q_2$.

**Approach 2: Factorise $\hat{q}$ and $q_2$, and calculate the geodesic distance between two measure-preserving mappings**

We seek an appropriate analogue of the polar factorisation of a real valued function. Suppose we can find $\bar{q} \in R(q_2)$ such that for every $q \in R(q_2)$ there exists a measure-preserving mapping $s : \Omega \to \Omega$ such that $q = \bar{q} \circ s$. Then we can find measure-preserving mappings $\hat{s}, s_2 : \Omega \to \Omega$ such that $\hat{q} = \bar{q} \circ \hat{s}$ and $q_2 = \bar{q} \circ s_2$. Assuming $\Omega$ is compact, we use the methods of Brenier [**2**] to calculate the "geodesic distance" between $\hat{s}$ and $s_2$ in the set of measure-preserving mappings from $\Omega$ to itself. (In fact there may be no shortest path between $\hat{s}$ and $s_2$, so the concept of a generalised path is needed to solve the problem.) The value we obtain is the distance between $\hat{q}$ and $q_2$.

The difficulties which remain with this approach are firstly to determine when such a $\bar{q} \in R(q_2)$ exists, and secondly that the factorisation of $\hat{q}$ and $q_2$ into the composition of $\bar{q}$ with a measure-preserving mapping may not be unique.

### 3.3 Extension to vector valued quantities

Thus far we have restricted attention to the case when $q$ represents a scalar valued quantity such as rainfall. However we may wish to calculate forecast error for wind velocity, which is vector valued. We indicate the difficulties involved and the work required to justify our formulation of error decomposition for a vector valued function $q$.

The results in Section 3.1 were crucially dependent on inequalities satisfied by the increasing rearrangement of a scalar valued function. The concept of increasing does not make sense for vector valued functions. However, an increasing scalar valued function is the derivative of a convex function. Replacing derivative with gradient, this is a concept which is well defined for vector valued functions, therefore we seek a rearrangement of a vector valued function equal to the gradient of a convex function. If $\Omega \subset \mathbb{R}^n$ is bounded, and $f_0 : \Omega \to \mathbb{R}^n$ is an integrable function, then the main result of McCann [**13**] yields that there is a unique element $f_0^\# \in R(f_0)$ which is equal to the gradient of a convex function. $f_0^\#$ is called the *monotone rearrangement of $f_0$*. See Cullen and Douglas [**9**] for an example. However the monotone rearrangement does not satisfy an inequality analogous to (2) (see Brenier [**3**]), therefore we need another approach to showing that $\inf_{q \in R(q_2)} ||q_1 - q_2||_2$ is attained by some $\hat{q} \in R(q_2)$. This is work which remains to be done.

Section 3.2 detailed two approaches to calculating the distance between two functions which are rearrangements of each other. The second required the existence of a factorisation of the rearrangements. For $f_0$ as above satisfying a technical condition, Burton and Douglas [**7**] showed

that for every $f \in R(f_0)$, there exists a measure-preserving mapping $s : \Omega \to \Omega$ such that $f = f_0^{\#} \circ s$. This composition of functions is known as the *polar factorisation of* $f$. Moreover Burton and Douglas proved that if $f_0^{\#}$ is injective, then the measure-preserving mapping is unique. If the restriction on the existence of the polar factorisation can be removed, then we may calculate the distance between two rearrangements according to the second approach. This result would be appropriate to the calculation of displacement error for wind velocity, for example.

# 4 Discussion and future directions

In this section we make some preliminary remarks about how our mathematical formulation of forecast error decomposition might be implemented numerically. We then discuss two drawbacks inherent in the idea of forecast error decomposition, and outline an alternative strategy.

## 4.1 Numerical implementation of forecast error decomposition

In the previous section it was shown that the concepts of polar factorisation of a scalar or vector valued function, and distance between measure-preserving mappings, were crucial to a mathematical formulation of forecast error decomposition. We indicate some relevant numerical results in the literature. Let $\Omega \subset \mathbb{R}^n$ be open, bounded and convex, and let $f : \Omega \to \mathbb{R}^n$ be an integrable function. If $f$ satisfies a non-degeneracy condition (see Brenier [**3**] for details, but note that this entails that $f$ does not take a uniform value on any set of positive size), then Benamou [**1**] details a scheme for finding the polar factorisation of $f$. This works for both scalar and vector valued functions. It remains to find a scheme which gives a polar factorisation for functions which fail the non-degeneracy condition. As regards calculating the geodesic distance between two elements in the set of measure-preserving mappings, Brenier [**4**] describes a discrete version of this problem. Polar factorisation plays an important role in this scheme (see [**4**] for details).

## 4.2 Defects in the proposed strategy

The precise mathematical formulation of forecast error decomposition which we considered in Section 2 is valid for any bounded set $\Omega \subset \mathbb{R}^n$. However it may give answers which we do not want because of limitations in the original idea. An error which is essentially an error in qualitative features may be penalised as though it were a displacement error. We illustrate two specific shortcomings.

(i) Suppose that we wish to decompose rainfall forecast error over the United Kingdom. It is possible that $\hat{q}$, the rearrangement of the actual rainfall distribution $q_2$ which is nearest to the forecast $q_1$, is achieved by rearranging rainfall from a wet area (e.g. Scotland) to a dry area a long distance away (e.g. the South East of England). Figure 2 illustrates this with sketch diagrams of forecast, actual and rearranged actual distributions for a hypothetical example. The displacement error term will reflect the rearrangement of values. However the error is really one of qualitative features. We can lessen the impact of this problem by restricting the region over which we perform the forecast error decomposition to those which are "meteorologically similar".

(ii) Suppose that the forecast is qualitatively wrong in two (disjoint) areas, but that the forecast in area 1 is close to the actual distribution in area 2, and that the predicted distribution in area 2 is close to the actual values in area 1. Such a situation is sketched (for imaginary data)

in Figure 3. However $\hat{q}$, the rearrangement of the true distribution $q_2$ which is closest to the forecast $q_1$, will have swapped values from area 1 to area 2 and vice versa, see the rearranged true distribution of Figure 3. The distance between $\hat{q}$ and $q_2$ in $R(q_2)$ is large in comparison to $||q_1 - \hat{q}||_2$, so the error has been interpreted as one of displacement. Whilst this sort of situation may not be common, it needs to be addressed because we require robustness in any measure of forecast error. One possibility would be to minimise the combined quantity of qualitative difference and displacement error over $q \in R(q_2)$, that is consider the problem

$$\inf_{q \in R(q_2)} \{||q_1 - q||_2 + \text{dist } (q, q_2)\},$$

where $\text{dist}(q, q_2)$ denotes the distance between $q$ and $q_2$ in $R(q_2)$. However we would need to show that such a problem is well posed.

### 4.3 An alternative strategy

An alternative approach to the problem of evaluating the error between forecast a $q_1$ and an actual distribution $q_2$ is to use a compressible velocity field to advect $q_1$ to $q_2$ directly, and calculate the an action integral analogous to (6). Now define the error between $q_1$ and $q_2$ to be the Least Action amongst all such vector fields. Results of McCann [**12**] suggest this may be possible. However it would be necessary to demonstrate that this procedure is well posed. Moreover this "measure of error" is not descriptive of how the forecast went wrong, which is an attractive feature of forecast error decomposition.

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## 7 Figure Captions

1. Sketch diagrams of the forecast, true and displaced true distributions for a hypothetical example.
2. Sketch diagrams of the forecast, actual and rearranged actual rainfall distributions for a hypothetical example which illustrates that the closest rearrangement of the actual distribution to the forecast may rearrange values from a wet area to a dry area a long distance away.
3. Sketch diagrams of the forecast, true and rearranged true distributions for a hypothetical example showing that errors in qualitative features may be interpreted as displacement errors.

FORECAST

TRUE

DISPLACED TRUE

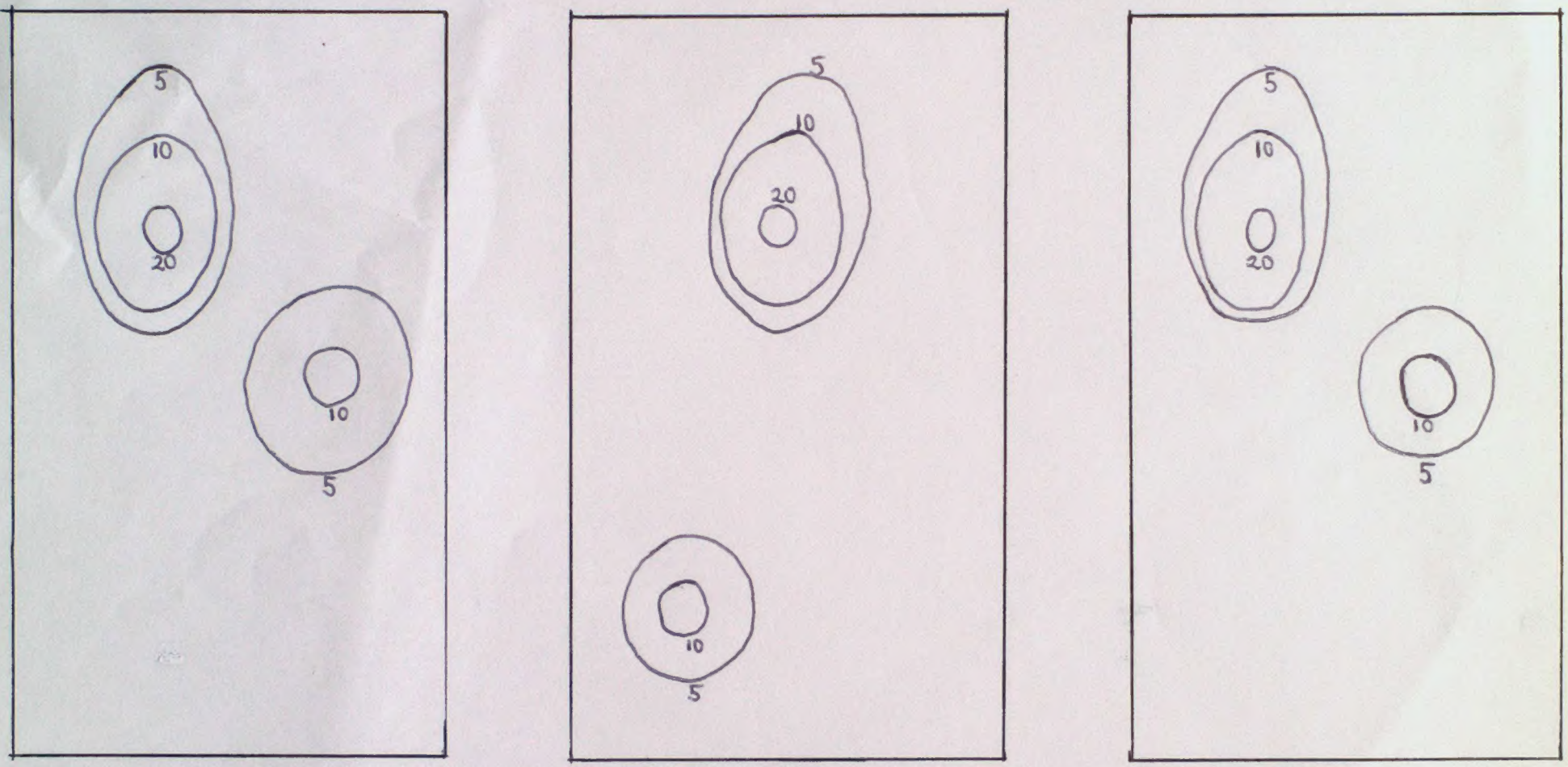


FIGURE 1

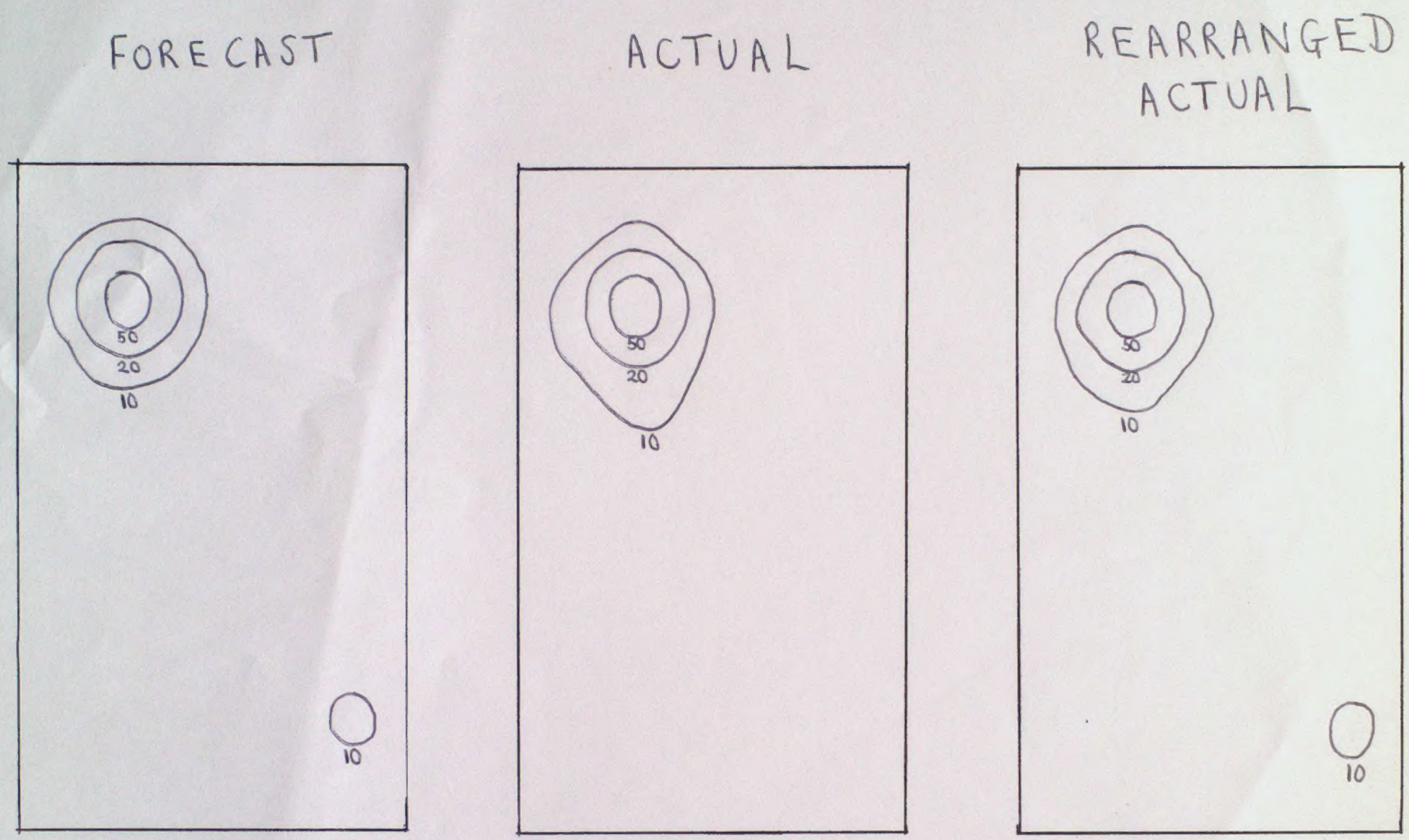
FORECAST
50
20
10
10
ACTUAL
50
20
10
REARRANGED
ACTUAL
50
20
10
10

FIGURE 2

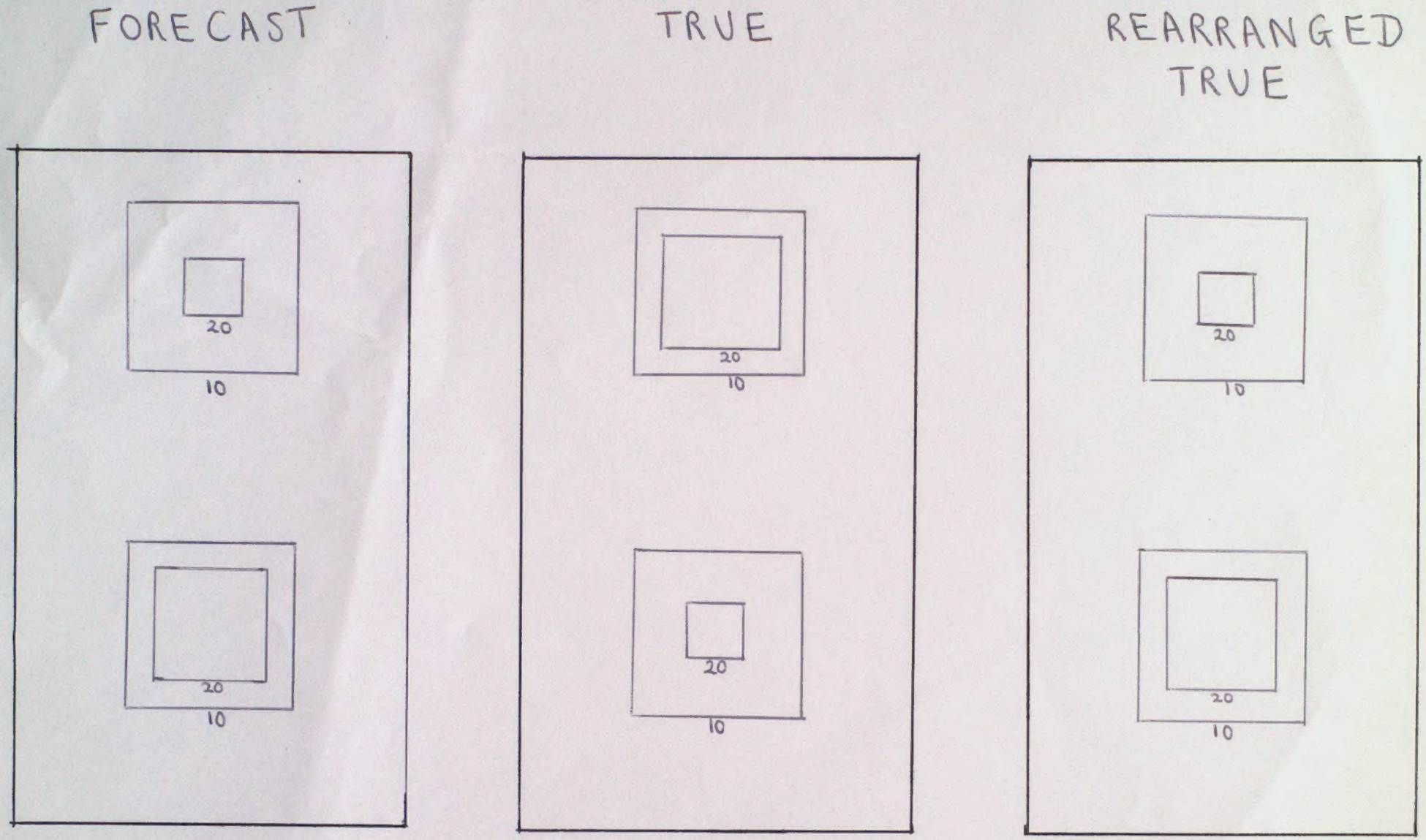
FORECAST
TRUE
REARRANGED TRUE
20
10
20
10
20
10
20
10
20
10
20
10

FIGURE 3